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Topological Charge Quantization via Path Integration: An Application of the Kustaanheimo–Stiefel Transformation[1]

Akira Inomata,[2] Georg Junker,[3] and Raj Wilson[4]

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The unified treatment of the Dirac monopole, the Schwinger monopole, and the Aharonov–Bohm problem by Barut and Wilson is revisited via a path integral approach. The Kustaanheimo–Stiefel transformation of space and time is utilized to calculate the path integral for a charged particle in the singular vector potential. In the process of dimensional reduction, a topological charge quantization rule is derived, which contains Dirac's quantization condition as a special case.

"Everything that is made beautiful and fair and lovely is made for the eye of one who sees."
Jelaluddin Rumi, *Mathnawi* [I, 2383]

1. INTRODUCTION

In recent years, the Kustaanheimo–Stiefel (KS) transformation, originally developed for celestial mechanics,[(1)] has become a useful tool in quantum mechanics. The idea of Kustaanheimo and Stiefel was to study the Kepler problem by applying a time scaling together with a coordinate transformation. The KS transformation consists of a space mapping $\mathbb{R}^4 \mapsto \mathbb{R}^3$ and a conversion of the physical time into a new timelike parameter. The KS transformation as a combined transformation of space and time has been applied for the unified algebraic treatment of the spherical top, the

[1] Dedicated to Prof. Asim O. Barut on the occasion of his 65th birthday.
[2] Department of Physics, State University of New York at Albany, Albany, New York 12222.
[3] Institut für Theoretische Physik I, Universität Erlangen-Nürnberg, Staudtstr. 7, D-91058 Erlangen, Germany.
[4] Division of Mathematics, University of Texas, San Antonio, Texas 78249.

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hydrogen atom, and the four-dimensional harmonic oscillator by Ikeda and Miyachi,[2] and to solve the Schrödinger equation for the hydrogen atom by Boiteux and Lucas.[3] Barut, Schneider, and Wilson[4] have related the KS transformation to the $SO(4)$ group of the internal quantum motion of composite objects in the framework of the $SO(4, 2)$ infinite-component theory. The Hartmann potential (or a ring-shaped potential), suitable for certain molecular chains, is another example which has been solved with the help of the KS transformation by Kibler and Negadi.[5]

The idea of applying the KS transformation to Feynman's path integral for the hydrogen atom was pointed out by Barut and Inomata as early as 1976, and discussed more seriously by Barut and Duru in 1978.[6] Finding a path integral solution for the hydrogen atom was indeed a long-standing problem. The work of Duru and Kleinert[7] in 1979 is the earliest one published. By using the KS transformation formally to the path integral, they obtained an integral representation of the Coulomb propagator. Subsequently, Ho and Inomata[8] discussed a way to implement the KS transformation in explicit path integration and to derive the Green function. There are some difficulties in implementing the KS transformation in explicit path integration, which are often ignored in formal application. The new position-dependent time parameter considered by Kustaanheimo and Stiefel is the same as or at least proportional to what is historically called the *eccentric anomaly*. It can be integrated only along a classical Kepler orbit. In path integration, we have to take account of contributions from all possible paths and have no fixed orbit to define the eccentric anomaly uniquely. Furthermore, due to the surjective nature of the spatial part of the KS transformation, it is difficult to define the Jacobian properly for the path integral. Recently, Castrigiano and Stärk[9] have provided a well-founded stochastic formulation of the KS transformation for Wiener integrals, resolving such difficulties.

Despite the fact that the KS transformation helped to make a breakthrough in solving the hydrogen atom problem by path integration, it has been found soon later that use of the KS mapping of coordinates is not essential in finding the path integral solution for this problem. An alternative treatment uses polar coordinates.[10] Even in spaces of constant curvature, Barut, Inomata, and Junker[11,12] have been able to solve the Coulomb problem by path integration without the KS transformation. What remains essential is only the KS *time* transformation. These observations may give an impression that the KS space transformation has no important role in path integration. However, as we shall discuss in this paper, there are cases where the KS transformation still *plays* a vital role. Application of the KS transformation to the path integral for the magnetic monopole problem leads to charge quantization very naturally without invoking the single-valuedness argument. This is in contrast to the standard treatment based on the Schrödinger equation in which the charge quantization condition arises only through the single-valuedness requirement on the wave functions.[13] Even in a path integral treatment,[14] if the KS transformation is not used, charge quantization has to be assumed in order to perform path integration.

The purpose of this paper is to present a unified path-integral approach to the quantization problem for an electric charge moving in the field of a Dirac monopole, a Schwinger monopole, or an infinitely thin flux tube. This is a path integral version of the unified group-theoretical treatment of the same problem by Barut and Wilson.[15] The approach is similar to the one applied to the Kaluza–Klein monopole problem.[16]

In Section 2, we incorporate the KS space and time transformation into the path integral by closely following the work of Castrigiano and Stärk.[9] In Section 3, we discuss the unified description of the singular vector potential of the field of the Dirac monopole, the Schwinger monopole, and a more general monopole. To implement the KS transformation, we deal with the energy-dependent Green function expressed in terms of a four-dimensional path integral of an object which we call the promotor. In Section 4, we calculate the angular integration of the promotor. Here we use the rules for time-sliced path integrals[17] to absorb the effect of the vector potential into the angular part of the kinetic energy, thus reducing the angular path integral to the path integral on the group manifold of $SU(2)$ which has been well studied.[18,19] The dimensional reduction of the promotor in four dimensions to the one in three dimensions, which is indeed a consequence of the KS coordinate mapping, naturally leads to a topological charge quantization condition which includes Dirac's and Schwinger's as special cases. Section 6 is devoted to evaluating the remaining radial path integral for two exactly solvable examples. The first example is an electric charge moving in the field of a dyon with a Dirac tail, for which we obtain a closed-form expression for the radial Green function, and the second one is an electric charge in the field of a Schwinger monopole, for which we find the radial propagator in closed form.

2. THE KS TRANSFORMATION IN PATH INTEGRALS

The KS transformation consists of the surjective map of space coordinates,

$$\Phi: \quad \mathbb{R}^4 \mapsto \mathbb{R}^3, \qquad \mathbf{u} \mapsto \mathbf{r} = \Phi[\mathbf{u}] \tag{2.1}$$

and a path-dependent time transformation defined in (2.5) below. For cartesian coordinates $\mathbf{r}=(x, y, z)$ and $\mathbf{u}=(u^1, u^2, u^3, u^4)$, the mapping (2.1) may be realized by

$$\begin{aligned} x &= 2(u^1u^3 - u^2u^4) \\ y &= 2(u^1u^4 + u^2u^3) \\ z &= (u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 \end{aligned} \tag{2.2}$$

The realization is not unique. Our choice is slightly different from the one used by Castrigiano and Stärk,[9] but has a small advantage that when expressed in polar variables the KS coordinates $\mathbf{u}=\mathbf{u}(u, \theta, \phi, \psi)\in\mathbb{R}^4$ are related to the usual spherical polar variables $\mathbf{r}=\mathbf{r}(r, \theta, \phi)\in\mathbb{R}^3$ by

$$\Phi: \begin{array}{l} u^1 = u\cos(\theta/2)\cos((\phi+\psi)/2) \\ u^2 = u\cos(\theta/2)\sin((\phi+\psi)/2) \\ u^3 = u\sin(\theta/2)\cos((\phi-\psi)/2) \\ u^4 = u\sin(\theta/2)\sin((\phi-\psi)/2) \end{array} \mapsto \begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array} \tag{2.3}$$

with $r=u^2$. Here we have set $u=|\mathbf{u}|$ and $r=|\mathbf{r}|$. The ranges of the variables are $u\geqslant 0$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, and $-2\pi\leqslant\psi<2\pi$. Obviously, Φ maps a circle in $\mathbb{R}^4$ parametrized by the angle ψ into one point on $\mathbb{R}^3$.

Let us denote by $\mathscr{C}^4_{\mathbf{u}'}$ the space of open-ended paths in $\mathbb{R}^4$ starting at $\mathbf{u}'$ and by $\mathscr{C}^3_{\mathbf{r}'}$ the space of open-ended paths in $\mathbb{R}^3$ starting at $\mathbf{r}'=\Phi[\mathbf{u}']$, respectively. Then the mapping (2.1) induces the transformation of paths

$$K:\quad \mathscr{C}^4_{\mathbf{u}'}\mapsto\mathscr{C}^3_{\mathbf{r}'},\qquad \mathbf{u}(s)\mapsto\Phi[\mathbf{u}(s)],\qquad s\in[0,\infty) \tag{2.4}$$

Here s is a timelike variable parametrizing paths in $\mathbb{R}^4$. For the parametrization of paths in $\mathbb{R}^3$, we introduce the path-dependent parameter

$$t = t_u(\sigma) = 4\int_0^\sigma ds\,|\mathbf{u}(s)|^2 \tag{2.5}$$

which we identify with the physical time. This time transformation together with the map (2.4) leads to the following combined transformation of paths:

$$TK:\quad \mathscr{C}^4_{\mathbf{u}'}\mapsto\mathscr{C}^3_{\mathbf{r}'},\qquad \mathbf{r}(t)=\Phi[\mathbf{u}(\sigma_u(t))] \tag{2.6}$$

where $\sigma_u(t)$ is the inverse of $t_u(\sigma)$. Castrigiano and Stärk[9] have proved the following equality of Wiener integrals:

$$\int_{\mathscr{C}^3_{\mathbf{r}'}} dW^3[\mathbf{r}]\,F[\mathbf{r}] = \int_{\mathscr{C}^4_{\mathbf{u}'}} dW^4[\mathbf{u}]\,F[TK\mathbf{u}] \tag{2.7}$$

which is valid for an arbitrary functional $F[\mathbf{r}]$ defined on $\mathscr{C}^3_{\mathbf{r}'}$. In the above, $dW^3[\mathbf{r}]$ and $dW^4[\mathbf{u}]$ are the (unconditional) Wiener measures on $\mathscr{C}^3_{\mathbf{r}'}$ and $\mathscr{C}^4_{\mathbf{u}'}$, respectively.

For the Hamiltonian $\hat{H}=[\mathbf{p}-(e/c)\,\mathbf{A}(\mathbf{r})]^2/2m+V(\mathbf{r})$, the Euclidean propagator can be represented as a path integral by the Feynman–Kac–Itô formula[20]

$$\begin{aligned} \langle\mathbf{r}''|\,e^{-\tau\hat{H}}\,|\mathbf{r}'\rangle = \int_{\mathscr{C}^3_{\mathbf{r}'}} dW^3[\mathbf{r}]\,\delta(\mathbf{r}''-\mathbf{r}(\tau)) \\ \times\exp\left\{-\int_0^\tau dt\left[V(\mathbf{r}(t))-\frac{ie}{\hbar c}\mathbf{A}(\mathbf{r}(t))\cdot\dot{\mathbf{r}}(t)\right]\right\} \end{aligned} \tag{2.8}$$

Since the paths involved in (2.8) are all constrained by the δ-function to end at a fixed point $\mathbf{r}''$ in a time interval τ, application of the relation (2.7) into (2.8) is not immediately meaningful. The relation of Castrigiano and Stärk takes the propagator into a function which describes contributions from all the paths that arrive at the point $\mathbf{u}''$ in various time periods. In other words, (2.7) does not transform the propagator in three dimensions into a propagator in four dimensions. It establishes an equality between a set of propagators in three dimensions and the corresponding set of propagators in four dimensions.

The Green function defined by $G(\mathbf{r}'', \mathbf{r}'; E)=\langle\mathbf{r}''|\,(E-\hat{H})^{-1}\,|\mathbf{r}'\rangle$ for $\operatorname{Im} E>0$ can be put into the integral form

$$G(\mathbf{r}'', \mathbf{r}'; E) = \frac{1}{i\hbar}\int_0^\infty d\tau\, P^{(3)}(\mathbf{r}'', \mathbf{r}'; \tau) \tag{2.9}$$

where

$$P^{(3)}(\mathbf{r}'', \mathbf{r}'; \tau) = \langle\mathbf{r}''|\,e^{(i/\hbar)\tau(E-\hat{H})}\,|\mathbf{r}'\rangle \tag{2.10}$$

which we call the "three-dimensional promotor." By the Wick rotation $i\tau\to\tau$, the promotor as a function of real time may be formally converted into a Euclidean promotor in analogy to the Euclidean propagator, and can be expressed as a Wiener integral. Notice that the Green function $G(\mathbf{r}'', \mathbf{r}'; E)$ is a collection of all the promotors for the paths commencing from $\mathbf{r}'$ and ending at $\mathbf{r}''$ in various time intervals. It is indeed appropriate to apply the relation (2.7) to the set of promotors and equate the Green function in three dimensions to the Green function in four dimensions. Thus, again using the Feynman–Kac–Itô formula, we can express the Green function as

$$G(\mathbf{r}'', \mathbf{r}'; E) = -\frac{1}{4}\int_0^\infty d\sigma \int_{-2\pi}^{2\pi} d\psi'' \int_{\mathscr{C}^4_{\mathbf{u}'}} dW^4[\mathbf{u}]\, \delta(\mathbf{u}'' - \mathbf{u}(\sigma))$$
$$\times \exp\left\{4\int_0^\sigma ds \left[|\mathbf{u}(s)|^2 \left[E - V(\Phi[\mathbf{u}(s)])\right] + \frac{ie}{\hbar c}\mathbf{A}(\Phi[\mathbf{u}(s)]) \cdot \frac{d}{ds}\Phi[\mathbf{u}(s)]\right]\right\} \tag{2.11}$$

The angular integral over ψ'', which appeared in the process of converting the three-dimensional δ-function into the four-dimensional one, has the role of projecting the promotor in four dimensions into the one in three dimensions.(8)

Going back to real time by making the inverse Wick rotation $\sigma \to i\sigma$, we write the Green function (2.9) in the form

$$G(\mathbf{r}'', \mathbf{r}'; E) = \frac{1}{4i\hbar}\int_0^\infty d\sigma \int_{-2\pi}^{2\pi} d\psi''\, P^{(4)}(\mathbf{u}'', \mathbf{u}'; \sigma) \tag{2.12}$$

with the "four-dimensional promotor" given by Feynman's path integral in four dimensions,

$$P^{(4)}(\mathbf{u}'', \mathbf{u}'; \sigma) = \int_{\mathbf{u}' = \mathbf{u}(0)}^{\mathbf{u}'' = \mathbf{u}(\sigma)} \mathscr{D}\mathbf{u} \exp\left[\frac{i}{\hbar} W\right] \tag{2.13}$$

where the effective action has the form

$$W = \int_0^\sigma ds \left\{\frac{m}{2}\left(\frac{d\mathbf{u}}{ds}\right)^2 + \frac{e}{c}\mathbf{A}(\Phi[\mathbf{u}]) \cdot \frac{d}{ds}\Phi[\mathbf{u}] + 4\,|\mathbf{u}|^2[E - V(\Phi[\mathbf{u}])]\right\} \tag{2.14}$$

In the usual time-slicing with $\varepsilon = \sigma/N$, this path integral is well defined by

$$P^{(4)}(\mathbf{u}'', \mathbf{u}'; \sigma) = \lim_{N\to\infty} \int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4} \prod_{j=1}^{N-1} d^4\mathbf{u}_j \prod_{j=1}^{N} \left[\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^2 e^{(i/\hbar)W_j}\right] \tag{2.15}$$

with the short-time action,

$$W_j = \frac{m}{2\varepsilon}(\Delta\mathbf{u}_j)^2 + \frac{e}{c}\mathbf{A}(\overline{\Phi[\mathbf{u}_j]}) \cdot \Delta\Phi[\mathbf{u}_j] + 4\hat{u}_j^2\varepsilon[E - V(\Phi[\mathbf{u}_j])] \tag{2.16}$$

We have used the standard notation $\mathbf{u}_j = \mathbf{u}(j\varepsilon)$, $\Delta\mathbf{u}_j = \mathbf{u}_j - \mathbf{u}_{j-1}$ and similarly for $\mathbf{r}_j = \Phi[\mathbf{u}_j]$. To this time-sliced path integral the usual rules of path integration can be applied.(17) For example, in path integration, the value of the vector potential $\mathbf{A}$ at the arithmetic midpoint $\bar{\mathbf{r}}_j \equiv \overline{\Phi[\mathbf{u}_j]} = \frac{1}{2}(\Phi[\mathbf{u}_j] + \Phi[\mathbf{u}_{j-1}])$ must be used. For the scalar potential $V(\mathbf{r})$, however, the midpoint rule does not have to be applied. Here we have chosen the geometric mean $\hat{u}_j^2 = u_j u_{j-1}$ for the radial variable, $u_j = |\mathbf{u}_j|$.

Now our problem is to calculate a four-dimensional path integral for the promotor. The promotor is an object which has no immediate physical interpretation but helps to evaluate, particularly with the relation (2.7), the physically meaningful Green function (2.9).

3. SINGULAR VECTOR POTENTIALS

The vector potential of a monopole which has N singularity lines extending from the origin to infinity along the unit vectors $\mathbf{e}_k$ is given by(21,22)

$$\mathbf{A}^{(N)}(\mathbf{r}) = \sum_{k=1}^{N} \mathbf{A}_k(\mathbf{r}) \tag{3.1}$$

where

$$\mathbf{A}_k(\mathbf{r}) = g_k \frac{\mathbf{r} \times \mathbf{e}_k}{r(r - \mathbf{r}\cdot\mathbf{e}_k)} \tag{3.2}$$

This potential leads to a magnetic field of the form

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{g}{r^2}\mathbf{e}_r \tag{3.3}$$

which is nothing but the B-field due to a magnetic charge,

$$g = \sum_{k=1}^{N} g_k \tag{3.4}$$

sitting at the origin. If all the singularities are physically indistinguishable, then we must have $g_k = g/N$.

For a unified treatment of the Dirac monopole, the Schwinger monopole, and the Aharonov–Bohm problem on the basis of the dynamical group $SO(4, 2)$, Barut and Wilson(15) introduced a set of indices (α, β) to parametrize the singular vector potential (3.1) as

$$\mathbf{A}^{(\alpha,\beta)}(\mathbf{r}) = g\frac{\alpha + \beta z/r}{r^2 - z^2}(y\mathbf{e}_x - x\mathbf{e}_y) = -g\frac{\alpha + \beta\cos\theta}{r\sin\theta}\mathbf{e}_\phi, \qquad \alpha, \beta \in \mathbb{R} \tag{3.5}$$

The Dirac monopole field, the Schwinger monopole field, and the Aharonov–Bohm field are specified by $(\pm 1, 1)$, $(0, 1)$, and $(1, 0)$, respectively. In this section we shall slightly generalize the singular potential (3.2), and prepare it in the form appropriate for path integration.

In polar coordinates, the partial vector potential can be expressed as

$$\mathbf{A}_k(\mathbf{r}) = g_k \frac{1+\cos\vartheta_k}{r\sin\vartheta_k}\mathbf{e}_{\varphi_k} \tag{3.6}$$

where $\vartheta_k = \arccos(\mathbf{r}\cdot\mathbf{e}_k/r)$. Following Barut and Wilson, let us define a vector potential

$$\mathbf{A}_k^{(\alpha,\beta)}(\mathbf{r}) = -g_k \frac{\alpha+\beta\cos\vartheta_k}{r\sin\vartheta_k}\mathbf{e}_{\varphi_k} \tag{3.7}$$

which is related to (3.6), if $g_k = \beta g$, by a gauge transformation

$$\mathbf{A}_k^{(\alpha,\beta)}(\mathbf{r}) \to \mathbf{A}_k^{(\alpha',\beta)}(\mathbf{r}) = \mathbf{A}_k^{(\alpha,\beta)}(\mathbf{r}) + \nabla\Lambda^{(\alpha'-\alpha)}(\mathbf{r}) \tag{3.8}$$

with the gauge function $\Lambda^{(\alpha)}(\mathbf{r}) = \alpha g\varphi_k$. In order for the singularity to lie only in the positive direction of $\mathbf{e}_k$, it is necessary to assume that $\alpha=\beta$.

The monopole sitting at the origin indeed creates a real singularity in configuration space, but the singular tails of the vector potential are not real. They are coordinate singularities. As Wu and Yang[23] suggested, the space around the monopole may be divided into singularity-free regions, say, region R_a and region R_b. The vector potential $\mathbf{A}_k^{(\beta,\beta)}(\mathbf{r})$ is appropriate in region R_a, if region R_a is defined by a region of $\mathbb{R}^3$ from which a sharp cone-shaped region enclosing the singularity along the $\mathbf{e}_k$-direction is cut out. Similarly, the potential $\mathbf{A}_k^{(-\beta,\beta)}(\mathbf{r})$ may be used to define region R_b in such a manner that the potential is everywhere regular. In the overlapping region, the two potentials are related by a gauge transformation of the form (3.8).

By an appropriate rotation of the coordinate frame, it is always possible to make one of the singularities to lie along the positive z-axis. Let $\mathbf{e}_1$ be in the positive z-direction. Then, we have

$$\mathbf{A}_1^{(\beta,\beta)}(\mathbf{r}) = \beta g_1 \frac{1+\cos\theta}{r\sin\theta} \tag{3.9}$$

Therefore, the Dirac monopole potentials in regions R_a and R_b are described by

$$\mathbf{A}_a^{(D)} = \mathbf{A}_1^{(1,1)} \quad\text{and}\quad \mathbf{A}_b^{(D)} = \mathbf{A}_1^{(-1,1)} \tag{3.10}$$

respectively. Each region of the Dirac monopole ($N=1$) is simply connected. If we find the propagator in region R_a, then we can get the propagator in region R_b by a gauge transformation.

While the nonsingular regions of the Dirac monopole are simply connected, those of the Schwinger monopole ($N=2$) are doubly connected. For the Schwinger monopole, we divide each doubly connected region into two singularity-free sectors. Expressing the vector potential for region R_a as

$$\mathbf{A}_a^{(S)} = \mathbf{A}_1^{(1/2,1/2)} + \mathbf{A}_2^{(-1/2,1/2)} \tag{3.11}$$

we describe the vector potentials in the first and the second sector by $\mathbf{A}_1^{(1/2,1/2)}$ and $\mathbf{A}_2^{(-1/2,1/2)}$, respectively. Similarly, for region R_b, we write

$$\mathbf{A}_b^{(S)} = \mathbf{A}_1^{(-1/2,1/2)} + \mathbf{A}_2^{(1/2,1/2)} \tag{3.12}$$

As in the Dirac monopole case, if the propagator in the first sector is found, then the propagator in the second sector can be obtained by a gauge transformation. However, this time, the contributions from the two sectors in one region have to be added to define the final propagator for the region. The propagator in one region is also related to the propagator in the other region via a gauge transformation.

In general, to cover a monopole carrying N singular tails, we have to consider the surrounding space N times connected. We divide one region into N sectors (sheets) and define in each sector one of the N vector potentials of (3.1). All of the vector potentials defined in separate sectors are connected by gauge transformations. Therefore, it is sufficient to calculate the propagator for a particular sector. The propagator of an electrically charged particle in the vector potential having N singular tails is given by

$$K_a(\mathbf{r}'',\mathbf{r}';\tau) = \sum_{k=1}^{N} K_a^{(k)}(\mathbf{r}'',\mathbf{r}';\tau) \qquad \text{in } R_a \tag{3.13}$$

and

$$K_b(\mathbf{r}'',\mathbf{r}';\tau) = \sum_{k=1}^{N} K_b^{(k)}(\mathbf{r}'',\mathbf{r}';\tau) \qquad \text{in } R_b \tag{3.14}$$

As far as the system is three-dimensional, the propagator must be single-valued or double-valued. Hence no extra phase factor need be introduced before each partial propagator.

In the case of the Aharonov–Bohm field,[24] the vector potential is given by

$$\mathbf{A}^{(AB)} = \sum_{k=1}^{\infty} \mathbf{A}_k^{(\alpha,0)} \tag{3.15}$$

which may be viewed as the limiting case of $N \to \infty$. The space of this potential cannot be separated into a finite number of singularity-free sectors. Since $\beta = 0$, the **B**-field of (3.1) vanishes for $r^2 \neq z^2$. In contrast to the monopole cases, the flux line along the z-axis is a real singularity which makes the configuration space infinitely connected, i.e., $\mathbb{R}^3 \backslash z$-axis. Therefore, in path integration, it is necessary to take the nontrivial topological structure into consideration. The monopole charge αg should be interpreted as the flux confined in the infinitesimally thin tube. The gauge transformation (3.8), as it alters the value of the flux in the tube, is a singular gauge transformation.

4. ANGULAR PATH INTEGRATION

If the Euler variables are used, with the singular vector potential (3.5), the short-time action (2.16) (we will consider central potentials $V(\mathbf{r}) = V(r)$ only) takes the form

$$W_j = \frac{m}{2\varepsilon}(\Delta u_j)^2 + \frac{m\hat{u}_j^2}{\varepsilon}\left[1 - \cos\frac{\Theta_j}{2}\right]$$
$$- q\hbar(\alpha + \beta\cos\bar{\theta}_j)\,\Delta\phi_j + 4\hat{u}_j^2\varepsilon[E - V(\hat{u}_j)] \tag{4.1}$$

As usual, we have used the notation $\bar{\theta}_j = (\theta_j + \theta_{j-1})/2$, $\Delta u_j = u_j - u_{j-1}$, and so on. Furthermore, we have set $q = eg/(\hbar c)$. We have also introduced the angle Θ_j by

$$\cos(\Theta_j/2) = \cos(\theta_j/2)\cos(\theta_{j-1}/2)\cos((\Delta\phi_j + \Delta\psi_j)/2)$$
$$+ \sin(\theta_j/2)\sin(\theta_{j-1}/2)\cos((\Delta\phi_j - \Delta\psi_j)/2) \tag{4.2}$$

In path integration, we have also to note that $d^4\mathbf{u} = (1/8)\,u^3 \sin\theta\, du\, d\theta\, d\phi\, d\psi$.

Now we calculate the angular part of the path integral (2.15) with action (4.1). Except for the term $\beta q\cos\bar{\theta}_j\,\Delta\phi_j$, the angular part of the action is identical with that of a point particle moving freely on the unit sphere S^3 in $\mathbb{R}^4$. Since the manifold S^3 is homeomorphic with the group manifold of $SU(2)$, the angular path integration for $\beta = 0$ is the same as the path integral over the group $SU(2)$, which has been well studied in recent years.[17-19] In the following we shall show, using the rules of time-sliced path integration,[17] that the term proportional to β in (4.1) can be absorbed in the angle Θ_j and that the resultant path integral can be handled by the group-theoretical technique.

The angular terms in (4.1) may be written explicitly in the form

$$\frac{m\hat{u}_j^2}{\varepsilon}\cos\frac{\Theta_j}{2} + \beta q\hbar\cos\bar{\theta}_j\,\Delta\phi_j =$$
$$\frac{m\hat{u}_j^2}{\varepsilon}\cos\frac{\theta_j}{2}\cos\frac{\theta_{j-1}}{2}\left[\cos\left(\frac{\Delta\phi_j + \Delta\psi_j}{2}\right)\right.$$
$$\left. + \frac{2\beta q\hbar\varepsilon}{m\hat{u}_j^2}\left(\frac{\Delta\phi_j + \Delta\psi_j}{2} - \Delta\psi_j/2\right)\right]$$
$$+ \frac{m\hat{u}_j^2}{\varepsilon}\sin\frac{\theta_j}{2}\sin\frac{\theta_{j-1}}{2}\left[\cos\left(\frac{\Delta\phi_j - \Delta\psi_j}{2}\right)\right.$$
$$\left. - \frac{2\beta q\hbar\varepsilon}{m\hat{u}_j^2}\left(\frac{\Delta\phi_j - \Delta\psi_j}{2} + \Delta\psi_j/2\right)\right] \tag{4.3}$$

With the aid of the approximation trick,[25]

$$\cos\Delta\varphi \pm \delta\,\Delta\varphi = \cos(\Delta\varphi \mp \delta) + \delta^2/2 + O(\delta^3)$$

the right-hand side may be put into the form

$$\frac{m\hat{u}_j^2}{\varepsilon}\cos(\tilde{\Theta}_j/2) + \frac{4\beta^2 q^2\hbar^2\varepsilon}{2m\hat{u}_j^2}\cos(\Delta\theta_j/2) - \beta q\hbar\,\Delta\psi_j\cos(\Delta\theta_j/2)$$

where

$$\cos(\tilde{\Theta}_j/2) = \cos(\theta_j/2)\cos(\theta_{j-1}/2)\cos((\Delta\phi_j + \Delta\tilde{\psi}_j)/2)$$
$$+ \sin(\theta_j/2)\sin(\theta_{j-1}/2)\cos((\Delta\phi_j - \Delta\tilde{\psi}_j)/2) \tag{4.4}$$

Here we have introduced the shifted angle $\Delta\tilde{\psi}_j = \Delta\psi_j - 4\beta q\hbar\varepsilon/(m\hat{u}_j^2)$. This angle is introduced only for making the group-theoretical calculation more tractable. In reality, we are not changing variables in the path integral. Path integration can be performed with the original variable ψ.

Moreover, we use the valid approximations $\varepsilon\cos(\Delta\theta_j/2) = \varepsilon + O(\varepsilon^2)$ and $\Delta\psi_j\cos(\Delta\theta_j/2) = \Delta\psi_j + O(\varepsilon^{3/2})$ to write the short-time action (4.1) in the equivalent form

$$W_j = \frac{m}{2\varepsilon}(\Delta u_j)^2 + \frac{m\hat{u}_j^2}{\varepsilon}[1 - \cos(\tilde{\Theta}_j/2)] - \frac{2\beta^2 q^2\hbar^2\varepsilon}{m\hat{u}_j^2}$$
$$+ 4\hat{u}_j^2\varepsilon[E - V(\hat{u}_j)] + \beta q\hbar\,\Delta\psi_j - \alpha q\hbar\,\Delta\phi_j \tag{4.5}$$

It is interesting to note that the extra centrifugal potential appearing in the above action is identical to the one that was introduced by Zwanziger[26] in an *ad hoc* manner in order to restore the dynamical group $SO(4)$ for the dyonium problem. Here the $SO(4)$ symmetry is realized as a geometrical symmetry in $\mathbb{R}^4$ via the KS transformation and as a consequence of the absorption of the vector potential in the kinetic energy. No *ad hoc* modifications are exercised for generating the $SO(4)$ symmetry.

The angular part of the action is now similar to that of a particle moving freely on the $SU(2)$-group manifold parametrized by Euler angles $(\phi, \theta, \tilde{\psi})$. The explicit harmonic analysis for the $\tilde{\Theta}$ terms gives[19]

$$\begin{aligned}
\exp\left\{-\frac{im\hat{u}_j^2}{\hbar\varepsilon}\cos(\tilde{\Theta}_j/2)\right\}
&=\frac{2i\hbar\varepsilon}{m\hat{u}_j^2}\sum_{2l=0}^{\infty}\sum_{\mu,\nu=-l}^{l}(2l+1)\,I_{2l+1}\left(\frac{m\hat{u}_j^2}{i\hbar\varepsilon}\right)\\
&\quad\times e^{-i\mu\Delta\phi_j}e^{-i\nu\Delta\tilde{\psi}_j}d^l_{\mu\nu}(\theta_j)\,d^{l*}_{\mu\nu}(\theta_{j-1})\\
&=\frac{2i\hbar\varepsilon}{m\hat{u}_j^2}\sum_{2l=0}^{\infty}\sum_{\mu,\nu=-l}^{l}(2l+1)\,I_{2l+1}\left(\frac{m\hat{u}_j^2}{i\hbar\varepsilon}\right)\exp\left\{\frac{4i\nu\beta q\hbar}{m\hat{u}_j^2}\right\}\\
&\quad\times e^{-i\mu\Delta\phi_j}e^{-i\nu\Delta\psi_j}d^l_{\mu\nu}(\theta_j)\,d^{l*}_{\mu\nu}(\theta_{j-1})
\end{aligned}\tag{4.6}$$

In the above, the first sum is to be taken over all integer and half-odd-integer values for the representation label l of the unitary irreducible representations of $SU(2)$. In the last step of (4.6) we have substituted back the shifted angle $\Delta\tilde{\psi}$ by the original one $\Delta\psi$. The function $I_\nu(z)$ is the modified Bessel function of the first kind and $d^l_{\mu\nu}(\theta)$ are the Wigner polynomials.

The angular path integration can be carried out with the help of the orthogonality relation of the Wigner polynomials

$$\int_0^\pi d\theta\,\sin\theta\,d^l_{\nu\mu}(\theta)\,d^{l'*}_{\nu'\mu'}(\theta)=\frac{2}{2l+1}\,\delta_{l,l'}\delta_{\nu,\nu'}\delta_{\mu,\mu'}\tag{4.7}$$

The explicit calculation yields

$$\begin{aligned}
P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)&=\sum_{2l=0}^{\infty}\sum_{\mu\nu=-l}^{l}\frac{2l+1}{4\pi^2}\,P_l(u'',u';\sigma)\\
&\quad\times e^{i(\alpha q+\mu)(\phi''-\phi')}e^{i(\beta q-\nu)(\psi''-\psi')}d^l_{\mu\nu}(\theta'')\,d^{l*}_{\mu\nu}(\theta')
\end{aligned}\tag{4.8}$$

where the remaining radial path integral is given by

$$\begin{aligned}
P_l(u'',u';\sigma)&=\frac{2}{u''u'}\lim_{N\to\infty}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{N-1}u_j\,du_j\prod_{j=1}^{N}\left[\frac{m}{i\hbar\varepsilon}\,I_{2l+1}\left(\frac{m\hat{u}_j^2}{i\hbar\varepsilon}\right)\right]\\
&\quad\times\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\varepsilon}(u_j^2+u_{j-1}^2)-\frac{2\beta q\hbar\varepsilon}{m\hat{u}_j^2}(\beta q\hbar-2\nu\hbar)\right.\right.\\
&\quad\left.\left.+4\hat{u}_j^2\varepsilon[E-V(\hat{u}_j^2)]\right]\right\}
\end{aligned}\tag{4.9}$$

5. CHARGE QUANTIZATION

Having completed the angular integration, we are now ready to perform the ψ''-integration in (2.12). We shall show that the projection from a circle in $\mathbb{R}^4$ onto one point in $\mathbb{R}^3$ gives rise to a charge quantization condition.

In order to perform the additional ψ''-integration we remind ourselves that

$$G(\mathbf{r}'',\mathbf{r}';E)=\frac{1}{4i\hbar}\int_0^\infty d\sigma\int_{-2\pi}^{2\pi}d\psi''\,P^{(4)}(\mathbf{u}'',\mathbf{u}';\sigma)\tag{5.1}$$

With the result (4.8), the ψ''-integration becomes as simple as

$$\frac{1}{4}\int_{-2\pi}^{2\pi}d\psi''\,e^{i(\beta q-\nu)\psi''}=\pi\delta_{\nu,\beta q}\tag{5.2}$$

Since ν can only be an integer or a half-odd-integer, the quantity βq has to be quantized in order to get a nonzero transition amplitude:

$$\beta q\equiv\beta\frac{eg}{\hbar c}=\frac{n}{2},\qquad n=0,\pm1,\pm2,\ldots\tag{5.3}$$

Demanding the indistinguishability of the singular lines, we can let $\beta=1/N$. Thus we are naturally led to the well-known charge-quantization condition of Dirac for $N=1$. Note that we have not invoked any single-valuedness argument. It is obtained by requiring a nonvanishing transition amplitude during the projection from $\mathbb{R}^4$ onto $\mathbb{R}^3$. This is indeed a topological quantization procedure.

For $N=2$ or $\beta=1/2$, we obtain Schwinger's quantization condition. In other words, the Schwinger monopole may be seen as carrying two singular tails. More generally, if the monopole has N tails, then the charge must be quantized by $q = Nn/2$.

In the Aharonov–Bohm problem, since $\beta = 0$, the topological charge quantization procedure does not at all apply. In this case, the quantity αq corresponds to the flux contained in the singularity along the z-axis. Again, the topological quantization procedure does not quantize the flux.

In concluding this section we write down the Green function (5.1) as

$$G(\mathbf{r}'', \mathbf{r}'; E) = \frac{1}{i\hbar}\int_0^\infty d\sigma \sum_{l=|\beta q|}^{\infty} \sum_{\mu=-l}^{l} P_l(u'', u'; \sigma)\, Y^{(\alpha)}_{\beta q,l,\mu}(\theta'', \phi'')\, Y^{(\alpha)*}_{\beta q,l,\mu}(\theta', \phi') \tag{5.4}$$

Here we have defined

$$Y^{(\alpha)}_{\beta q,l,\mu}(\theta, \phi) = \left[\frac{2l+1}{4\pi}\right]^{1/2} d^l_{\mu,\beta q}(\theta)\, \bar{e}^{i(\alpha q + \mu)\phi} \tag{5.5}$$

which is a generalization of the monopole harmonics of Wu and Yang.[23] Note that in (5.4) the l-sum is now either over all integer or all half-odd-integer angular momenta depending on whether βq is integer or half-odd-integer, respectively. The radial promotor is given by the path integral (4.9) with $\nu = \beta q$, $r'' = u''^2$, and $r' = u'^2$.

6. RADIAL PATH INTEGRATION: TWO EXACT RESULTS

6.1. A Charge in the Field of a Dyon

As the first example we consider a particle of mass m with an electric charge e which moves in the field of a dyon. The dyon carries an electric charge Ze as well as a magnetic charge g with a Dirac tail along the positive or negative z-axis. In this case, the parameters for the vector potential (3.1) are fixed to $\alpha = \pm 1$ and $\beta = 1$. In addition, we have the scalar potential $V(r) = -Ze^2/r$. The Hamiltonian reads

$$\hat{H}_1 = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^{(\pm 1,1)}(\mathbf{r})\right)^2 - \frac{Ze^2}{r} \tag{6.1}$$

Here, (1, 1) for R_a and $(-1, 1)$ for R_b. The solution can be expressed in either R_a or R_b. The solutions found in the two regions are connected by a gauge transformation, and hence physically equivalent.

The remaining radial path integral that has to be computed is

$$P_l(u'', u'; \sigma) = \frac{2}{u''u'} \lim_{N\to\infty} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{N-1} u_j\, du_j \prod_{j=1}^{N}\left[\frac{m}{i\hbar\varepsilon} I_{2l+1}\left(\frac{m\hat{u}_j^2}{i\hbar\varepsilon}\right)\right]$$
$$\times \prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\varepsilon}(u_j^2 + u_{j-1}^2) - \frac{4q^2\hbar^2}{2m\hat{u}_j^2}\varepsilon + 4E\hat{u}_j^2\varepsilon + 4Ze^2\varepsilon\right]\right\} \tag{6.2}$$

This radial path integral is identical in form with that of a harmonic oscillator of frequency $\omega = \sqrt{-8E/m}$ in an inverse-square potential. This path integral is explicitly solvable.[27] Note that the Coulomb potential appears as an additive constant. The result of path integration is

$$P_l(u'', u'; \sigma) = \frac{2m\omega}{i\hbar u''u' \sin(\omega\sigma)} e^{4iZe^2\sigma/\hbar} \exp\left\{\frac{im\omega}{2\hbar}(u''^2 + u'^2)\cot(\omega\sigma)\right\}$$
$$\times I_{2\gamma+1}\left(\frac{m\omega}{i\hbar} u''u' \csc(\omega\sigma)\right) \tag{6.3}$$

where we have set $\gamma = [(l + 1/2)^2 - q^2/\hbar^2]^{1/2} - 1/2$.

With (6.3) the remaining σ-integration (5.1) for the Green function can be performed. This integration has to be carefully done so as to obtain the retarded Green function. For this reason, we go back to the Euclidean time by the Wick rotation $i\sigma \to \sigma$. If the Green function is written in the form

$$G(\mathbf{r}'', \mathbf{r}'; E) = \sum_{l=|q|}^{\infty} G_l(r'', r'; E) \sum_{\mu=-l}^{l} Y^{(\alpha)}_{q,l,\mu}(\theta'', \phi'')\, Y^{(\alpha)*}_{q,l,\mu}(\theta', \phi') \tag{6.4}$$

with $\alpha = 1$ for R_a and $\alpha = -1$ for R_b, the radial Green function is given by

$$G_l(r'', r'; E) = -\frac{1}{\hbar}\int_0^\infty d\sigma\, P_l(u'', u'; -i\sigma)$$
$$= -\frac{2m\omega}{\hbar^2 u''u'}\int_0^\infty d\sigma \frac{e^{4Ze^2\sigma/\hbar}}{\sinh(\omega\sigma)} \exp\left\{-\frac{m\omega}{2\hbar}(u''^2 + u'^2)\coth(\omega\sigma)\right\}$$
$$\times I_{2\gamma+1}\left(\frac{m\omega u''u'}{\hbar \sinh(\omega\sigma)}\right) \tag{6.5}$$

This integral can be converted into a form found in the integral table[28] if the substitution $\sinh(x) = 1/\sinh(\omega\sigma)$ is made, the result being

$$G_l(r'', r'; E) = -\frac{2}{\hbar\omega}\frac{\Gamma(-\kappa+\gamma+1)}{\Gamma(2\gamma+2)}\frac{1}{r''r'}$$
$$\times\left\{M_{\kappa,\gamma+1/2}\left(\frac{m\omega}{\hbar}r'\right)W_{\kappa,\gamma+1/2}\left(\frac{m\omega}{\hbar}r''\right)\Theta(r''-r')\right.$$
$$\left.+M_{\kappa,\gamma+1/2}\left(\frac{m\omega}{\hbar}r''\right)W_{\kappa,\gamma+1/2}\left(\frac{m\omega}{\hbar}r'\right)\Theta(r'-r'')\right\} \qquad (6.6)$$

where $\kappa = 2Ze^2/\hbar\omega$, $M_{a,b}(z)$, and $W_{a,b}(z)$ are the Whittaker functions, and $\Theta(z)$ is the unit-step function. The poles of the Γ-function in the numerator of (6.6) give rise to the discrete energy spectrum of the system, and the residues at the poles give us the energy eigenfunctions. These calculations are similar to those for the hydrogen atom, for which we refer to the literature.[17] The Green function (6.6) can easily be generalized to that for the dyonium problem with an additional scalar inverse-square potential.[29]

6.2. A Charge in the Field of the Schwinger Monopole

The second example we wish to consider is that of a charge e moving in the field of Schwinger's monopole. In this case, the parameters of the vector potential (3.1) are set to $\alpha = \pm 1/2$ and $\beta = 1/2$. We assume no additional scalar potential, i.e., $V(r) = 0$. As has been mentioned earlier, the configuration space around the Schwinger monopole must be divided into four simply connected sectors. Let us specify the first and the second sector in region R_a by (1/2, 1/2) with $\phi \in [0, 2\pi)$ and (−1/2, 1/2) with $\phi \in [2\pi, 4\pi)$, respectively. The sectors in R_b are specified by reversing the sign of α. The Hamiltonian is

$$\hat{H}_2^{(\pm)} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^{(\pm 1/2, 1/2)}(\mathbf{r})\right)^2 \qquad (6.7)$$

As far as calculation is concerned, the radial path integral for this system can be treated as a special case of the one calculated in the previous section. We simply let $Z = 0$, and hence $\kappa = 0$. Since the vector potential regular in each sector can be related to those in other sectors by gauge transformations, it is sufficient to calculate the path integral for the first sector. The full Green function in R_a is, however, the sum of the contributions from the two sectors:

$$G(\mathbf{r}'', \mathbf{r}'; E) = \sum_{l=|q/2|}^{\infty} G_l(r'', r'; E) \sum_{\alpha=\pm 1/2} \sum_{\mu=-l}^{l} Y^{(\alpha)}_{q/2,l,\mu}(\theta'', \phi''+\pi-2\pi\alpha)$$
$$\times Y^{(\alpha)*}_{q/2,l,\mu}(\theta', \phi'+\pi-2\pi\alpha) \qquad (6.8)$$

with the radial Green function,

$$G_l(r'', r'; E) = -\frac{2m}{\hbar^2}\frac{1}{\sqrt{r''r'}}\left\{I_{\gamma+1/2}\left(\frac{m\omega}{2\hbar}r'\right)K_{\gamma+1/2}\left(\frac{m\omega}{2\hbar}r''\right)\Theta(r''-r')\right.$$
$$\left.+I_{\gamma+1/2}\left(\frac{m\omega}{2\hbar}r''\right)K_{\gamma+1/2}\left(\frac{m\omega}{2\hbar}r'\right)\Theta(r'-r'')\right\} \qquad (6.9)$$

where $\gamma = [(l+1/2)^2 - q^2/4]^{1/2} - 1/2$. If the parameter κ vanishes, the Whittaker functions in (6.6) become modified Bessel functions of the first and third kind (denoted by $I_\nu(z)$ and $K_\nu(z)$), respectively.[30] Obviously, the Green function (6.8) has no poles; this electron–monopole system has no bound states.

Using the integral formula[31]

$$I_\nu(az)\,K_\nu(bz) = \int_0^\infty d\tau\, e^{-z^2\tau}(2\tau)^{-1}\, e^{-(a^2+b^2)/4\tau} I_\nu(ab/4\tau)$$

which is valid for $a < b$ and $\mathrm{Re}\, z^2 > 0$, we may obtain the Euclidean propagator if we set $z^2 = m\omega^2/8\hbar$. After performing the inverse Wick rotation $\tau \to i\tau$, we obtain the real time propagator in R_a for the Schwinger monopole problem:

$$K(\mathbf{r}'', \mathbf{r}', \tau) = \sum_{l=|q/2|}^{\infty} K_l(r'', r'; \tau) \sum_{\alpha=\pm 1/2} \sum_{\mu=-l}^{l} Y^{(\alpha)}_{q/2,l,\mu}(\theta'', \phi''+\pi-2\pi\alpha)$$
$$\times Y^{(\alpha)*}_{q/2,l,\mu}(\theta', \phi'+\pi-2\pi\alpha) \qquad (6.10)$$

where the radial propagator is given by

$$K_l(r'', r'; \tau) = \frac{m}{i\hbar\tau}\frac{1}{\sqrt{r''r'}}\exp\left\{\frac{im}{2\hbar\tau}(r''^2+r'^2)\right\} I_{\gamma+1/2}\left(\frac{mr''r'}{i\hbar\tau}\right) \qquad (6.11)$$

From this expression we can obtain the normalized wave functions by utilizing the integral formula

$$\int_0^\infty dk\, k \exp\left\{-\frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}\tau\right\} J_{\gamma+1/2}(kr'')\, J_{\gamma+1/2}(kr')$$
$$= \frac{m}{i\hbar\tau}\exp\left\{\frac{im}{2\hbar\tau}(r''^2+r'^2)\right\} I_{\gamma+1/2}\left(\frac{mr''r'}{i\hbar\tau}\right)$$

Here $J_\nu(z)$ stands for the ordinary Bessel function. The spectral decomposition for the propagator may be made as

$$K(\mathbf{r}'', \mathbf{r}', \tau) = \int_0^\infty dE \exp\{-(i/\hbar) E\tau\}$$

$$\times \sum_{\alpha = \pm 1/2} \sum_{l=|q/2|}^{\infty} \sum_{\mu=-l}^{l} \Psi^{(\alpha)}_{E,l,\mu}(\mathbf{r}'') \Psi^{(\alpha)}_{E,l,\mu}(\mathbf{r}') \quad (6.12)$$

with the normalized energy eigenfunction,

$$\Psi^{(\alpha)}_{E,l,\mu}(\mathbf{r}) = \frac{\sqrt{m}}{\hbar} r^{-1/2} J_{\gamma+1/2}(\sqrt{2mE}\, r/\hbar)\, Y^{(\alpha)}_{q/2,l,\mu}(\theta, \phi + \pi - 2\pi\alpha) \quad (6.13)$$

The result (6.13) for $\alpha = 1/2$ coincides with the solution obtained by Barut and Wilson.[15] The energy eigenfunction in R_b is given by

$$\Psi^{(\alpha)}_{E,l,\mu}(\mathbf{r}) = \frac{\sqrt{m}}{\hbar} r^{-1/2} J_{\gamma+1/2}(\sqrt{2mE}\, r/\hbar)\, Y^{(-\alpha)}_{q/2,l,\mu}(\theta, \phi + \pi - 2\pi\alpha) \quad (6.14)$$

7. CONCLUDING REMARKS

In this paper we have employed the Kustaanheimo–Stiefel transformation in formulating a unified path-integral treatment of the Dirac monopole, the Schwinger monopole, and more general monopoles. We have shown without resort to the single-valuedness requirement that a charge quantization condition can be derived naturally by dimensional reduction in the process of path integration. The projection from the four-dimensional space $\mathbb{R}^4$, where we have performed path integration, onto the configuration space $\mathbb{R}^3$ has led to the quantization condition. It is indeed a topological quantization. The N times connectedness of the space for singular vector potentials is reflected on the charge quantization formula $eg/\hbar c = Nn/2$. For the Dirac monopole ($N = 1$), this condition coincides with the well-known charge-quantization condition of Dirac. For the Schwinger monopole ($N = 2$), we obtained Schwinger's quantization condition. More generally, the formula applies for a charge around a monopole with N singular tails. We have also observed that the topological quantization procedure does not quantize the flux involved in the Aharonov–Bohm field.[24] Although the Aharonov–Bohm problem can be included in the present unified treatment, it is simpler to calculate Feynman's path integral for the propagator directly as has been done previously.[32]

This article is written as a contribution to the special issues of this journal in honor of the 65th birthday of Professor Asim O. Barut. Every one of the present authors has been highly inspired by Professor Barut's wit, courage, and scholarship. We would like to express our sincere appreciation and best wishes to him on this special occasion.

REFERENCES

1. P. Kustaanheimo and E. Stiefel, *J. Reine Angew. Math.* **218**, 204 (1965).
2. M. Ikeda and Y. Miyachi, *Math. Jpn.* **15**, 127 (1970).
3. M. M. Boiteux and M. Rene Lucas, *C. R. Acad. Sci. (Paris)* **274**, 867 (1972).
4. A. O. Barut, C. K. E. Schneider, and R. Wilson, *J. Math. Phys.* **20**, 2244 (1979).
5. M. Kibler and T. Negadi, *Theor. Chim. Acta* **66**, 31 (1984).
6. Seminar discussions in Munich (1976) and in Trieste (1978).
7. I. H. Duru and H. Kleinert, *Phys. Lett. B* **84**, 185 (1979); *Fortsch. Phys.* **30**, 401 (1982).
8. R. Ho and A. Inomata, *Phys. Rev. Lett.* **48**, 231 (1982).
9. D. P. L. Castrigiano and F. Stärk, *J. Math. Phys.* **30**, 2785 (1989).
10. A. Inomata, *Phys. Lett. A* **101**, 253 (1984).
11. A. O. Barut, A. Inomata, and G. Junker, *J. Phys. A* **20**, 6271 (1987).
12. A. O. Barut, A. Inomata, and G. Junker, *J. Phys. A* **23**, 1179 (1990).
13. J. J. Sakurai, *Modern Quantum Mechanics* (Benjamin/Cummings, Menlo Park, California, 1985).
14. H. Dürr and A. Inomata, *J. Math. Phys.* **26**, 2231 (1985).
15. A. O. Barut and R. Wilson, *Ann. Phys.* **164**, 223 (1985).
16. A. Inomata and G. Junker, *Phys. Rev. D* **43**, 1235 (1991).
17. A. Inomata, H. Kuratsuji, and C. C. Gerry, *Path Integrals and Coherent States of SU(2) and SU(1, 1)* (World Scientific, Singapore, 1992).
18. G. Junker and A. Inomata, in M. C. Gutzwiller, A. Inomata, J. R. Klauder, and L. Streit, eds., *Path Integrals from meV to MeV* (World Scientific, Singapore, 1986), p. 315.
19. M. Böhm and G. Junker, *J. Math. Phys.* **28**, 1978 (1987).
20. See, e.g., G. Roepstorff, *Pfadintegrale in der Quantenphysik* (Vieweg, Braunschweig, Germany, 1991), p. 114.
21. Yu. D. Usachev, *Sov. J. Part. Nucl.* **11**, 94 (1973).
22. A. O. Barut, *J. Phys. A* **11**, 2073 (1978).
23. T. T. Wu and C. N. Yang, *Phys. Rev. D* **12**, 3845 (1975); *Nucl. Phys. B* **107**, 365 (1976).
24. Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
25. A. Inomata and V. A. Singh, *J. Math. Phys.* **19**, 2318 (1978).
26. D. Zwanziger, *Phys. Rev.* **176**, 1480 (1968).
27. D. Peak and A. Inomata, *J. Math. Phys.* **10**, 1422 (1969).
28. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980), formula 6.669 4.
29. H. Kleinert, *Phys. Lett. A* **116**, 185 (1986).
30. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, 1966), p. 305.
31. See Ref. 30, p. 98.
32. C. C. Bernido and A. Inomata, *J. Math. Phys.* **22**, 715 (1981).